L20 LMP with two-sided H_a

1. LMP test with two-sided H_a

   (1) Recall
   For $H_0: \theta = \theta_0$ versus $H_a: \theta \neq \theta_0$, with sample $X = (X_1, ..., X_n)$ let $g(X_1, .., X_n) = f''_{\theta^2}(X; \theta_0) - k_1 f(X; \theta_0) - k_2 f'_\theta(X; \theta_0)$. Then

   $$\phi(X) = \begin{cases} 1 & g(X_1, .., X_n; \theta_0) > 0 \\ r & g(X_1, .., X_n; \theta_0) = 0 \\ 0 & g(X_1, .., X_n; \theta_0) < 0 \end{cases} \quad \text{with } \beta_\psi(\theta_0) = \alpha \text{ and } [\beta_\phi(\beta_0)]'_\theta = 0$$

   is LMP test at θ_0 over

   $$\mathcal{T}_2 = \left\{\psi : \beta_\psi(\theta_0) \leq \alpha, [\beta_\psi(\theta_0)]'_\theta = 0 \text{ ad } [\beta_\psi(\theta)]''_{\theta^2} \text{ is continuous at } \theta_0\right\}.$$

   (2) Using U and V
   Population has continuous distribution. Let $U = \frac{f'_\theta(X; \theta_0)}{f(X; \theta_0)}$ and $V = \frac{f''_{\theta^2}(X; \theta_0)}{f(X; \theta_0)}$. Then

   $$\phi(U, V) = \begin{cases} 1 & V > k_1 + k_2 U \\ 0 & V < k_1 + k_2 U \end{cases} \quad \text{with } E_{\theta_0}[\phi(U, V)] = \alpha \text{ and } E_{\theta_0}[U\phi(U, V)] = 0$$

   is LMP test at θ_0 over all tests in $\mathcal{T}_2$.

   Proof. $\begin{cases} g(X) > 0 \\ g(X) < 0 \end{cases} \Longleftrightarrow \begin{cases} V > k_1 + k_2 U \\ V < k_1 + k_2 U \end{cases}$, $\quad \beta_\phi(\theta_0) = E_{\theta_0}[\phi(U, V)]$ and
   $[\beta_\phi(\theta_0)]'_\theta = \int \phi(x) \frac{f'_\theta(x; \theta_0)}{f(x; \theta_0)} f(x; \theta_0)\, dx = E_{\theta_0}[\phi(X)U(X)] = E_{\theta_0}[U\phi(U, V)]$.

   Ex1: Write $U = U(X; \theta_0)$ and $V = V(X; \theta_0)$. Then $V = U'_\theta(X; \theta_0) + U^2$
   $U'_\theta(X; \theta) = \left(\frac{f'_\theta}{f}\right)'_\theta = \frac{f(X; \theta) f''_{\theta^2}(X; \theta) - [f'_\theta(X; \theta)]^2}{f^2(X; \theta)} = V(X; \theta) - U^2(X; \theta)$.
   So $U'_\theta(X; \theta_0) = V - U^2$, i.e., $V = U'_\theta(X; \theta_0) + U^2$.

2. Case of $\begin{pmatrix} U \\ V \end{pmatrix}_{\theta_0} \stackrel{L}{=\!=} \begin{pmatrix} -U \\ V \end{pmatrix}_{\theta_0}$

   (1) Lemma 1
   If population pdf $f(-x; \theta_0) = f(x; \theta_0)$, $U(-x_1, .., -x_n; \theta_0) = -U(x_1, .., x_n; \theta_0)$ and $V(-x_1, .., -x_n; \theta_0) = V(x_1, .., x_n; \theta_0)$, then $\begin{pmatrix} U \\ V \end{pmatrix}_{\theta_0} \stackrel{L}{=\!=} \begin{pmatrix} -U \\ V \end{pmatrix}_{\theta_0}$

   Pf: $\begin{pmatrix} U(X_1, .., X_n \theta_0) \\ V(X_1, .., X_n; \theta_0) \end{pmatrix} \stackrel{L}{=\!=} \begin{pmatrix} U(-X_1, .., -X_n; \theta_0) \\ V(-X_1, .., -X_n; \theta_0) \end{pmatrix} = \begin{pmatrix} -U(X_1, .., X_n; \theta_0) \\ V(X_1, .., X_n; \theta_0) \end{pmatrix}$.

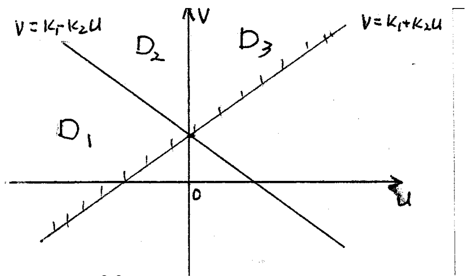
   (2) Lemma 2
   If $\begin{pmatrix} U \\ V \end{pmatrix}_{\theta_0} \stackrel{L}{=\!=} \begin{pmatrix} -U \\ V \end{pmatrix}_{\theta_0}$, then $\quad k_2 = 0 \Longleftrightarrow E_{\theta_0}[\phi(X)U(X)] = 0$

   Proof. $\Longrightarrow$
   When $k_2 = 0$, $\phi(U, V)$ does not depend on V. With $\begin{pmatrix} U \\ V \end{pmatrix}_{\theta_0} \stackrel{L}{=\!=} \begin{pmatrix} -U \\ V \end{pmatrix}_{\theta_0}$,
   $E_{\theta_0}[U\phi(U, V)] = E_{\theta_0}[-U\phi(-U, V)] = -E_{\theta_0}[U\phi(U, V)]$.

So $E_{\theta_0}[U\phi(U,\, V)] = 0$.

$\Longleftarrow$

$$\begin{aligned} E_{\theta_0}[U\phi(U,\, V)] &= \iint_{(u,\, v)} u\phi(u, v) f(u,\, v;\, \theta_0)\, ds = \iint_{v>k_1+k_2u} uf(u,\, v;\, \theta_0)\, ds \\ &= \iint_{D_1} + \iint_{D_2} + \iint_{D_3} uf(u,\, v;\, \theta_0)\, ds = \iint_{D_1} uf(u,\, v;\, \theta_0)\, ds. \end{aligned}$$



If $E_{\theta_0}[U\phi(U,\, V)] = 0$, then $\iint_{D_1} uf(u,\, v;\, \theta_0)\, ds = 0$. So the measure of D_1 is 0. Hence $k_2 = 0$.

(3) Theorem

Under the condition in Lemma 1, for $H_0 : \theta = \theta_0$ versus $H_a : \theta \neq \theta_0$,

$$\phi(V) = \begin{cases} 1 & V > c \\ 0 & V < c \end{cases} \quad \text{with } E_{\theta_0}[\phi(V)] = \alpha$$

is LMP test at θ_0 over all tests in $\mathcal{T}_2$.

3. An example

For $H_0 : \theta = 0$ versus $H_a : \theta \neq 0$ where θ is from a logistic population with pdf $f(x;\, \theta) = \frac{e^{-(x-\theta)}}{[1+e^{-(x-\theta)}]^2}$, show that the conditions of Lemma 1 hold.

(1) $f(x;\, 0) = \frac{e^{-x}}{(1+e^{-x})^2}$. $f(-x;\, 0) = \frac{e^{x}}{(1+e^{x})^2} \cdot \frac{e^{-2x}}{e^{-2x}} = \frac{e^{-x}}{(1+e^{-x})^2} = f(x;\, 0)$.

(2) From HW, $U(x_1, .., x_n;\, 0) = \sum_{i=1}^{n} \frac{1-e^{-x_i}}{1+e^{-x_i}}$. So $U(-x_1, .., -x_n;\, 0) = \sum_i \frac{1-e^{x_i}}{1+e^{x_i}} = \sum_i \frac{e^{-x_i}-1}{e^{-x_i}+1} = -\sum_i \frac{1-e^{-x_i}}{1+e^{-x_i}} = -U(x_1, .., x_n : 0)$

(3) $U(x_1, .., x_n;\, \theta) = \sum_i \frac{1-e^{-(x_i-\theta)}}{1+e^{-(x_i-\theta)}} \Longrightarrow U'_\theta(x_1, .., x_n;\, \theta) = \sum_i \frac{-2e^{-(x_i-\theta)}}{[1+e^{-(x_i-\theta)}]^2}$.

So $U'_\theta(x_1, .., x_n;\, 0) = \sum_i \frac{-2e^{-x_i}}{[1+e^{-x_i}]^2}$.

Thus $U'(-x_1, .., -x_n;\, 0) = \sum_i \frac{-2e^{x_i}}{[1+e^{x_i}]^2} = \sum_i \frac{-2e^{-x_i}}{[1+e^{-x_i}]^2} = U'_\theta(x_1, .., x_n;\, 0)$.

With $V(x_1, .., x_n;\, 0) = U'_\theta(x_1, .., x_n;\, 0) + U^2(x_1, .., x_n;\, 0)$,

$$\begin{aligned} V(-x_1, .., -x_n;\, 0) &= U'_\theta(-x_1, .., -x_n;\, 0) + U^2(-x_1, .., -x_n;\, 0) \\ &= U'_\theta(x_1, .., x_n;\, 0) + U^2(x_1, .., x_n;\, 0) = V(x_1, .., x_n;\, 0). \end{aligned}$$